ENGINEERING MATH - II

UNIT 4

INVERSE LAPLACE TRANSFORM

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Inverse LT of Standard functions

1. $\mathcal{L}^{-1}\left\{\frac{k}{s}\right\} = k$

2. $\mathcal{L}^{-1}\left\{\frac{1}{s^n}\right\} = \frac{1}{\Gamma(n)} t^{n-1}$ or $\frac{t^{n-1}}{(n-1)!}$

3. $\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$

4. $\mathcal{L}^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at}$

5. $\mathcal{L}^{-1}\left\{\frac{1}{s-c\ln b}\right\} = e^{c\ln b\, t} = b^{ct}$

6. $\mathcal{L}^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{1}{a}\sin(at)$

7. $\mathcal{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos(at)$

8. $\mathcal{L}^{-1}\left\{\frac{1}{s^2-a^2}\right\} = \frac{1}{a}\sinh(at)$

9. $\mathcal{L}^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh(at)$

10. $\mathcal{L}^{-1}\left\{\frac{1}{(s\mp a)^2+b^2}\right\} = \frac{e^{\pm at}}{b}\sin(bt)$

11. $\mathcal{L}^{-1}\left\{\frac{(s\mp a)}{(s\mp a)^2+b^2}\right\} = e^{\pm at}\cos(bt)$

$\swarrow$ n^{th} derivative

12. $\mathcal{L}^{-1}\left\{F^{n}(s)\right\} = (-1)^n t^n f(t)$

13. $\mathcal{L}^{-1}\left\{\int_s^\infty\int_s^\infty \cdots \int_s^\infty F(s)\,ds \cdots ds\,ds\right\} = \frac{f(t)}{t^n}$

14. $\mathcal{L}^{-1}\left\{\frac{F(s)}{s^n}\right\} = \int_0^\infty\int_0^\infty \cdots f(t) \cdots dt\,dt$

15. $\mathcal{L}^{-1}\left\{s\,F(s)\right\} = f'(t)$ iff $f(0)=0$

16. $\mathcal{L}^{-1}\left\{1\right\} = \delta(t)$

17. $\mathcal{L}^{-1}\left\{e^{-as}\right\} = \delta(t-a)$

18. $\mathcal{L}^{-1}\left\{\frac{e^{-as}}{s}\right\} = u(t-a)$

19. $\mathcal{L}^{-1}\left\{e^{-as}F(s)\right\} = f(t-a)\,u(t-a)$

20. $\mathcal{L}^{-1}\left\{e^{-as} f(a)\right\} = f(t)\,\delta(t-a)$

21. $\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s}}\right\} = \frac{1}{\sqrt{\pi t}}$

21. $\mathcal{L}^{-1}\left\{\frac{1}{s\sqrt{s}}\right\} = 2\sqrt{\frac{t}{\pi}}$

* 22. $\mathcal{L}^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\} = \int_0^t \frac{t}{2a} \sin at \, dt$

* 23. $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \frac{1}{2a} t \sin t$

24. $\mathcal{L}^{-1}\left\{\frac{1}{(s \mp a)^n}\right\} = \frac{e^{\pm at} t^{n-1}}{\Gamma(n)}$

* 25. $\mathcal{L}^{-1}\left\{\frac{s^2-a^2}{(s^2+a^2)^2}\right\} = e^{-at}$]

do property

26. $\mathcal{L}^{-1}\left\{\frac{1}{s \mp a - c\ln b}\right\} = e^{\pm at} b^{ct}$

27. $\mathcal{L}^{-1}\left\{\frac{1}{(s \mp a)^2 - b^2}\right\} = \frac{e^{\pm at}}{b} \sinh bt$

28. $\mathcal{L}^{-1}\left\{\frac{s \mp a}{(s \mp a)^2 - b^2}\right\} = e^{\pm at} \cosh bt$

I First Shifting Property of Inverse LT

If $\mathcal{L}^{-1}\{F(s)\} = f(t)$, $\mathcal{L}^{-1}\{F(s-a)\} = e^{at} f(t)$

1. $\mathcal{L}^{-1}\left\{ \frac{1}{s-4} \right\} = e^{4t}$

2. $\mathcal{L}^{-1}\left\{ \frac{1}{2s-3} \right\} = \frac{1}{2} \mathcal{L}\left\{ \frac{1}{s-3/2} \right\} = \frac{1}{2} e^{3t/2}$

make co-ef = 1

3. $\mathcal{L}^{-1}\left\{ \frac{8-6s}{16s^2+9} \right\} = \frac{1}{16} \mathcal{L}^{-1}\left\{ \frac{8}{s^2+\left(\frac{3}{4}\right)^2} - \frac{6s}{s^2+\left(\frac{3}{4}\right)^2} \right\}$

$= \frac{1}{2} \mathcal{L}^{-1}\left\{ \frac{1}{s^2+(3/4)^2} \right\} - \frac{3}{8} \mathcal{L}^{-1}\left\{ \frac{s}{s^2+(3/4)^2} \right\}$

$= \frac{1}{2} \frac{\sin\left(\frac{3}{4}t\right)}{3/4} - \frac{3}{8} \cos\left(\frac{3}{4}t\right)$

$= \frac{2}{3} \sin\left(\frac{3t}{4}\right) - \frac{3}{8} \cos\left(\frac{3t}{4}\right)$

4. $\mathcal{L}^{-1}\left\{ \frac{1}{(s-2)^3} \right\} = \frac{e^{2t}}{2} t^2$

5. $\mathcal{L}^{-1}\left\{ \frac{1}{s^2+4s+20} \right\} = \mathcal{L}^{-1}\left\{ \frac{1}{(s+2)^2+4^2} \right\} = \frac{e^{-2t} \sin 4t}{4}$

6. $\mathcal{L}^{-1}\left\{ \frac{s+3+1}{(s+3)^2+4} \right\} = e^{-3t}\left(\cos(2t) + \frac{\sin 2t}{2}\right)$

7. $\mathcal{L}^{-1}\left\{\frac{2}{s^3}+\frac{4}{s}\right\} = \frac{2t^2}{2}+4 = t^2+4$

8. $\mathcal{L}^{-1}\left\{\frac{1}{s+2}-\frac{2}{s-3}\right\} = e^{-2t}-2e^{3t}$

9. $\mathcal{L}^{-1}\left\{\frac{2s+5}{s^2+16}\right\} = 2\cos(4t)+\frac{5}{4}\sin(4t)$

10. $\mathcal{L}^{-1}\left\{\frac{1}{s^2+6s+18}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+3)^2+3^2}\right\} = \frac{e^{-3t}}{3}\sin 3t$

11. $\mathcal{L}^{-1}\left\{\frac{1}{s+2}+\frac{3}{2s+5}-\frac{4}{3s-2}\right\}$

$= e^{-2t}+\frac{3}{2}e^{-\frac{5}{2}t}-\frac{4}{3}e^{\frac{2}{3}t}$

12. $\mathcal{L}^{-1}\left\{\frac{5s-2}{s^2+4s+8}\right\} = \mathcal{L}^{-1}\left\{\frac{5(s+2)-12}{(s+2)^2+2^2}\right\}$

$= 5e^{-2t}\cos(2t)-\frac{12}{2}e^{-2t}\sin 2t$

$= e^{-2t}(5\cos 2t-6\sin 2t)$

13. $\mathcal{L}^{-1}\left\{\frac{s}{3s^2-2s-5}\right\} = \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{s}{s^2-\frac{2}{3}s-\frac{5}{3}}\right\}$

$= \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{s}{(s-1/3)^2-\frac{5}{3}-\frac{1}{9}}\right\} = \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{s}{(s-1/3)^2-\left(\frac{4}{3}\right)^2}\right\}$

$= \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{s-1/3}{(s-1/3)^2-(4/3)^2} + \frac{1/3}{(s-1/3)^2-(4/3)^2}\right\}$

$= \frac{1}{3}e^{t/3}\left(\cosh\left(\frac{4}{3}t\right) + \frac{1\times 3}{3\times 4}\sinh\left(\frac{4}{3}t\right)\right)$

$= \frac{1}{3}e^{t/3}\left(\cosh\left(\frac{4t}{3}\right) + \frac{1}{4}\sinh\left(\frac{4t}{3}\right)\right)$

II Method of Partial Fractions

Split the function using the method of partial fractions and then find its inverse.

Note: $\frac{1}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1}$

$\frac{1}{(x-1)(x+2)^2} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$

$\frac{1}{(x^2+1)(x-1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1}$

14. $\mathcal{L}^{-1}\left\{\frac{s+1}{(s-1)^2(s+2)}\right\}$

$$\frac{s+1}{(s-1)^2(s+2)} = \frac{A}{(s-1)^2} + \frac{B}{s-1} + \frac{C}{s+2}$$

$$s+1 = A(s+2) + B(s-1)(s+2) + C(s-1)^2$$

when $s=1$

$$2 = 3A \Rightarrow A = 2/3$$

when $s=-2$

$$-1 = 9C \Rightarrow C = -1/9$$

when $s=0$

$$1 = 2A - 2B + C$$

$$1 = \frac{4}{3} - 2B - \frac{1}{9}$$

$$\frac{-2}{9} = -2B \Rightarrow B = 1/9$$

$$\frac{s+1}{(s-1)^2(s+2)} = \frac{2/3}{(s-1)^2} + \frac{1/9}{s-1} \quad \frac{-1/9}{s+2}$$

$$\mathcal{L}^{-1}\left\{\frac{2/3}{(s-1)^2} + \frac{1/9}{s-1} \quad \frac{-1/9}{s+2}\right\}$$

$$= \frac{2}{3}e^t t + \frac{1}{9}e^t - \frac{1}{9}e^{-2t}$$

15. $\mathcal{L}^{-1}\left\{\frac{s+1}{(s^2+1)(s^2+4)}\right\}$

$$\frac{s+1}{(s^2+1)(s^2+4)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4}$$

$$s+1 = (As+B)(s^2+4) + (s^2+1)(Cs+D)$$

when $s = i$

$$1+i = (Ai+B)(3)$$

comparing real & imaginary coefficients

$3B = 1$ $\quad$ $3A = 1$

$B = 1/3$ $\quad$ $A = 1/3$

when $s = 2i$

$$2i+1 = (2Ci+D)(-3)$$

comparing real & imaginary coefficients

$2 = -6C$ $\quad$ $-3D = 1$

$C = -1/3$ $\quad$ $D = -1/3$

$$= \mathcal{L}^{-1}\left\{ \frac{1/3(s+1)}{s^2+1} - \frac{1/3(s+1)}{s^2+4} \right\}$$

$$= \frac{1}{3}(\cos t + \sin t) - \frac{1}{3}\left(\cos 2t + \frac{1}{2}\sin 2t\right)$$

$$= \frac{1}{3}\left(\cos t + \sin t - \cos 2t - \frac{1}{2}\sin 2t\right)$$

16. $$\mathcal{L}^{-1}\left\{ \frac{s}{s^4+s^2+1} \right\} = \mathcal{L}^{-1}\left\{ \frac{s}{s^4+2s^2+1-s^2} \right\}$$

$$= \mathcal{L}^{-1}\left\{ \frac{s}{(s^2+1)^2 - s^2} \right\} = \mathcal{L}^{-1}\left\{ \frac{s}{(s^2+1-s)(s^2+1+s)} \right\}$$

$$= \frac{1}{2}\mathcal{L}^{-1}\left\{ \frac{2s}{(s^2-s+1)(s^2+s+1)} \right\}$$

$$= \frac{1}{2}\mathcal{L}^{-1}\left\{ \frac{s^2+s+1-(s^2-s+1)}{(s^2-s+1)(s^2+s+1)} \right\}$$

$$= \frac{1}{2}\mathcal{L}^{-1}\left\{ \frac{1}{s^2-s+1} - \frac{1}{s^2+s+1} \right\}$$

$$= \frac{1}{2}\mathcal{L}^{-1}\left\{ \frac{1}{(s-1/2)^2+(\sqrt{3}/2)^2} - \frac{1}{(s+1/2)^2+(\sqrt{3}/2)^2} \right\}$$

$$= \frac{1}{2}\left(\frac{2}{\sqrt{3}} e^{t/2} \sin\left(\frac{\sqrt{3}t}{2}\right) - \frac{2}{\sqrt{3}} e^{-t/2} \sin\left(\frac{\sqrt{3}t}{2}\right) \right)$$

$$= \frac{2}{\sqrt{3}} \sin\left(\frac{\sqrt{3}t}{2}\right) \sinh\left(\frac{t}{2}\right)$$

17. $$\mathcal{L}^{-1}\left\{\frac{s}{s^4+4a^4}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{(s^2)^2+(2a^2)^2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s}{(s^2+2a^2)^2-(2as)^2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s}{(s^2+2a^2-2as)(s^2+2a^2+2as)}\right\}$$

$$= \frac{1}{4a}\mathcal{L}^{-1}\left\{\frac{(2as+s^2+2a^2)-(s^2+2a^2-2as)}{(s^2+2as+2a^2)(s^2-2as+2a^2)}\right\}$$

$$= \frac{1}{4a}\mathcal{L}^{-1}\left\{\frac{1}{s^2-2as+2a^2} - \frac{1}{s^2+2as+2a^2}\right\}$$

$$= \frac{1}{4a}\mathcal{L}^{-1}\left\{\frac{1}{(s-a)^2+a^2} - \frac{1}{(s+a)^2+a^2}\right\}$$

$$= \frac{1}{4a}\cdot\frac{1}{a}\left(e^{at}\sin at - e^{-at}\sin at\right)$$

$$= \frac{1}{2a^2}\sin at \sinh at$$

18. $\mathcal{L}^{-1}\left\{\frac{s^2}{(s^2+4)(s^2+9)}\right\}$

Let $s^2 = v$

$$\frac{v}{(v+4)(v+9)} = \frac{A}{v+4} + \frac{B}{v+9}$$

$$v = A(v+9) + B(v+4)$$

when $v = -4$

$$-4 = 5A \Rightarrow A = -4/5$$

when $v = -9$

$$-9 = -5B \Rightarrow B = 9/5$$

$$= \mathcal{L}^{-1}\left\{\frac{-4/5}{s^2+4} + \frac{9/5}{s^2+9}\right\}$$

$$= -\frac{4}{5} \times \frac{1}{2} \sin 2t + \frac{9}{5} \times \frac{1}{3} \sin 3t$$

$$= -\frac{2}{5} \sin 2t + \frac{3}{5} \sin 3t$$

III Multiplication by s and division by s

i) $\mathcal{L}^{-1}\{sF(s)\} = f'(t)$ if $f(0) = 0$

where $\mathcal{L}^{-1}\{F(s)\} = f(t)$

In General, if $f(0) = f'(0) = \cdots = f^{n-1}(0) = 0$, then

$$\mathcal{L}^{-1}\{s^n F(s)\} = \frac{d^n}{dt^n} f(t)$$

ii) If $\mathcal{L}^{-1}\{F(s)\} = f(t)$, then $\mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(t)\,dt$,

$$\mathcal{L}^{-1}\left\{\frac{F(s)}{s^2}\right\} = \int_0^t \int_0^t f(t)\,dt\,dt$$

19. $$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s+1)}\right\} = \int_0^t \int_0^t \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} dt\,dt$$

$$= \int_0^t \int_0^t e^{-t}\,dt\,dt = \int_0^t \left[-e^{-t}\right]_0^t dt$$

$$= \int_0^t -e^{-t} + 1\,dt = \left[e^{-t} + t\right]_0^t = e^{-t} - 1 + t$$

20. $$\mathcal{L}^{-1}\left\{\frac{s+2}{s^2(s+3)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s(s+3)} + \frac{2}{s^2(s+3)}\right\}$$

$$= \int_0^t \mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} dt + \int_0^t\int_0^t \mathcal{L}^{-1}\left\{\frac{2}{s+3}\right\} dt\, dt$$

$$= \int_0^t e^{-3t} dt + \int_0^t\int_0^t 2e^{-3t}\, dt\, dt$$

$$= \frac{-e^{-3t}}{3} + \frac{1}{3} + \int_0^t \frac{-2e^{-3t}}{3} + \frac{2}{3}\, dt$$

$$= \frac{-e^{-3t}}{3} + \frac{1}{3} + \frac{2}{3}\cdot\frac{1}{3}\left(e^{-3t} - 1\right)$$

$$= \frac{-e^{-3t}}{3} + \frac{1}{3} + \frac{2}{9}e^{-3t} - \frac{2}{9}$$

IV Inverse LT of Derivatives

$$\mathcal{L}\{t f(t)\} = -1 \frac{d}{ds} F(s)$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{d}{ds} F(s)\right\} = -t f(t)$$

or $$-\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{d}{ds} F(s)\right\} = f(t)$$

Similarly

$$\mathcal{L}^{-1}\left\{\frac{d^n}{ds^n} F(s)\right\} = (-1)^n t^n f(t)$$

21. $$\mathcal{L}^{-1}\left\{\ln\left(1+\frac{a^2}{s^2}\right)\right\} = f(t)$$

$$F(s) = \ln(s^2+a^2) - \ln(s^2)$$

$$\frac{d}{ds} F(s) = \frac{2s}{s^2+a^2} - \frac{2s}{s^2}$$

$$\mathcal{L}^{-1}\left\{\frac{d}{ds} F(s)\right\} = -t f(t)$$

$$= \mathcal{L}^{-1}\left\{\frac{2s}{s^2+a^2} - \frac{2}{s}\right\}$$

$$= 2\cos at - 2 = -t f(t)$$

$$\therefore f(t) = -\frac{2}{t}\cos at + \frac{2}{t}$$

22. $\mathcal{L}^{-1}\left\{ \ln\left(\frac{s^2+1}{s(s+1)}\right)\right\} = f(t)$

$$F(s) = \ln\left(\frac{s^2+1}{s(s+1)}\right)$$

$$F'(s) = \left(\frac{s(s+1)}{s^2+1}\right)\left(\frac{(2s)}{s(s+1)} + (s^2+1)(s^2+s)^{-2}(-1)(2s+1)\right)$$

$$F'(s) = \frac{1}{s^2+1}\left(2s + \frac{-(s^2+1)(2s+1)}{s(s+1)}\right)$$

$$= \frac{2s}{s^2+1} - \frac{(2s+1)}{s(s+1)}$$

$$\mathcal{L}^{-1}\{F'(s)\} = 2\cos t - \mathcal{L}^{-1}\left\{\frac{2}{s+1} + \frac{1}{s(s+1)}\right\}$$

$$= 2\cos t - 2e^{-t} - \int_0^t \mathcal{L}^{-1}\left(\frac{1}{s+1}\right) dt$$

$$= 2\cos t - 2e^{-t} + e^{-t} - 1 = -t f(t)$$

$$f(t) = \frac{-2\cos t + e^{-t} + 1}{t}$$

23. $$\mathcal{L}^{-1}\left\{\frac{s+2}{(s^2+4s+5)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{s+2}{((s+2)^2+1)^2}\right\}$$

$$= e^{-2t}\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\}$$

$$F(s) = \frac{1}{s^2+1} \Rightarrow F'(s) = \frac{-2s}{(s^2+1)^2}$$

$$= -\frac{e^{-2t}}{2}\mathcal{L}^{-1}\left\{\frac{-2s}{(s^2+1)^2}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{d}{ds}F(s)\right\} = -t f(t) = -t\,\mathcal{L}^{-1}\{F(s)\}$$

$$= \frac{-e^{-2t}}{2}(-t)\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= \frac{e^{-2t}}{2}t\sin t$$

Note: $$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \frac{1}{2a}t\sin at$$

$$F(s) = \frac{1}{s^2+a^2} \Rightarrow F'(s) = \frac{-2s}{(s^2+a^2)^2}$$

$$\mathcal{L}^{-1}\left\{\frac{F'(s)}{-2}\right\} = \frac{-t}{-2a}\sin at = \frac{t}{2a}\sin at$$

24. $\mathcal{L}^{-1}\left\{ s \ln\left(\frac{s-1}{s+1}\right)\right\} = f(t)$

$$F(s) = s \ln\left(\frac{s-1}{s+1}\right)$$

$$\frac{d}{ds}F(s) = \ln\left(\frac{s-1}{s+1}\right) + \frac{s(s+1)}{(s-1)}\left((1)\left(\frac{1}{s+1}\right) + \frac{(s-1)(-1)}{(s+1)^2}\right)$$

$$F'(s) = \ln\left(\frac{s-1}{s+1}\right) + \frac{s}{s-1} - \frac{s}{s+1}$$

$$\mathcal{L}^{-1}\{F'(s)\} = -t f(t)$$

$$\mathcal{L}^{-1}\left\{\ln\left(\frac{s-1}{s+1}\right)\right\} + \mathcal{L}^{-1}\left\{ 1 + \frac{1}{s-1} - 1 + \frac{1}{s+1}\right\}$$

$$= -\frac{1}{t}\mathcal{L}^{-1}\left\{\left(\frac{s+1}{s-1}\right)\left((1)\left(\frac{1}{s+1}\right) + \frac{(s-1)}{(s+1)^2}(-1)\right)\right\}$$

$$= -\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{1}{s-1} - \frac{1}{s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s-1} + \frac{1}{s+1}\right\}$$

$$= -\frac{1}{t}\left(e^{t} - e^{-t}\right) + e^{t} + e^{-t}$$

$$= \frac{1}{t^2}\left(e^{t} - e^{-t}\right) - \frac{1}{t}\left(e^{t} + e^{-t}\right)$$

$$= \frac{2}{t^2}\sinh t - \frac{2}{t}\cosh t$$

$$= \frac{2}{t^2}\left(\sinh t - t\cosh t\right)$$

25. $\mathcal{L}^{-1}\left\{ \tan^{-1}\left(\frac{s}{2}\right)\right\}$

$$= \frac{-1}{t}\mathcal{L}^{-1}\left\{ \frac{1/2}{1+\frac{s^2}{4}} \right\} = \frac{-1}{t}\mathcal{L}^{-1}\left\{ \frac{2}{s^2+2^2} \right\}$$

$$= \frac{-1}{t} \sin 2t$$

26. $\mathcal{L}^{-1}\left\{ \cot^{-1}\left(\frac{a}{s+b}\right)\right\} = f(t)$

$$\mathcal{L}^{-1}\{F'(s)\} = -t f(t)$$

$$\mathcal{L}^{-1}\{F'(s)\} = \mathcal{L}^{-1}\left\{ \frac{-1}{1+\left(\frac{a^2}{(s+b)^2}\right)} \cdot \frac{(a)(-1)}{(s+b)^2} \right\}$$

$$= \mathcal{L}^{-1}\left\{ \frac{a}{(s+b)^2+a^2} \right\}$$

$$f(t) = \frac{-1}{t} e^{-bt} \sin at$$

27. $\mathcal{L}^{-1}\left\{ \tanh^{-1}\left(\frac{2}{s}\right)\right\}$

$$= \frac{-1}{t}\mathcal{L}^{-1}\left\{ \frac{1}{1-\left(\frac{2}{s}\right)^2} \cdot 2 \cdot \frac{(-1)}{s^2} \right\}$$

$$= \frac{+1}{t} \mathcal{L}^{-1} \left\{ \frac{2}{s^2 - 2^2} \right\}$$

$$= \frac{1}{t} \sinh(2t)$$

28. $$\mathcal{L}^{-1} \left\{ \frac{1}{(s^2+a^2)^2} \right\} = \mathcal{L}^{-1} \left\{ \left(\frac{-1}{2s}\right)\left(\frac{-2s}{(s^2+a^2)^2}\right) \right\}$$

$$= \frac{-1}{2} \mathcal{L}^{-1} \left\{ \frac{1}{s} \frac{(-2s)}{(s^2+a^2)^2} \right\}$$

$$= \frac{-1}{2} \mathcal{L}^{-1} \left\{ \frac{1}{s} \frac{d}{ds} \left(\frac{1}{s^2+a^2} \right) \right\}$$

$$= \frac{-1}{2} \int_0^t \mathcal{L}^{-1} \left\{ \frac{d}{ds} \frac{1}{s^2+a^2} \right\} dt$$

$$= \frac{-1}{2} \int_0^t \frac{-t}{a} \sin at \, dt = \frac{1}{2a} \int_0^t t \sin at \, dt$$

$u = t$ $\quad v = \frac{-\cos at}{a}$

$du = dt$ $\quad dv = \sin at \, dt$

$$= \frac{1}{2a} \left[-\frac{t \cos at}{a} + \int_0^t \frac{\cos at}{a} dt \right]_0^t$$

$$= \frac{1}{2a} \left[-\frac{t \cos at}{a} + \frac{\sin at}{a^2} \right]_0^t$$

$$= \frac{1}{2a^2} \left(-at \cos at + \sin at \right)$$

V Inverse LT of Integrals

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\,ds$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\int_s^\infty F(s)\,ds\right\} = \frac{f(t)}{t}$$

29. $\mathcal{L}^{-1}\left\{\int_s^\infty \frac{a}{s^2+a^2}\,ds\right\} = \frac{1}{t}\sin at$

30. $\mathcal{L}^{-1}\left\{\int_s^\infty \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\,ds\right\} = \frac{1}{t}\left(\cos at - \cos bt\right)$

31. $\mathcal{L}^{-1}\left\{\int_s^\infty \ln\left(\frac{s+2}{s+1}\right)ds\right\}$

$$= \frac{-1}{t^2}\mathcal{L}^{-1}\left\{\left(\frac{s+1}{s+2}\right)\left(\frac{1}{s+1} + \frac{-(s+2)}{(s+1)^2}\right)\right\}$$

$$= \frac{-1}{t^2}\mathcal{L}^{-1}\left\{\frac{1}{s+2} - \frac{1}{s+1}\right\}$$

$$= \frac{-1}{t^2}\left(e^{-2t} - e^{-t}\right)$$

VI Second Shifting Property of Inverse LT

$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s)$, then

$\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)u(t-a)$, where

$$u(t-a) = \begin{cases} 0, & 0 < t < a \\ 1, & t \geq a \end{cases}$$

32. $\mathcal{L}^{-1}\left\{ \frac{2}{s} - \frac{2e^{-\pi s}}{s} + \frac{e^{-2\pi s}}{s^2+1} \right\}$

$= 2 - 2\mathcal{L}^{-1}\left\{ \frac{e^{-\pi s}}{s} \right\} + \mathcal{L}^{-1}\left\{ \frac{e^{-2\pi s}}{s^2+1} \right\}$

$= 2 - 2u(t-\pi) + u(t-2\pi)\sin(t-2\pi)$

The solution can be expressed as a discontinuous func.

$f(t) = f_1(t) + (f_2(t) - f_1(t))u(t-\pi) +$
$(f_2(t) - f_3(t))u(t-2\pi)$

$f(t) = \overset{f_1}{2} + (\overset{f_2}{0} - \overset{f_1}{2})u(t-\pi) + (\overset{f_3}{\sin(t-2\pi)} - \overset{f_2}{0})u(t-2\pi)$

$\sin(t-2\pi) = \sin t$ f_3

$$f(t) = \begin{cases} 2, & 0 < t < \pi \\ 0, & \pi < t < 2\pi \\ \sin t & t \geq 2\pi \end{cases}$$

33. $\mathcal{L}^{-1}\left\{ \dfrac{3s-12}{s^2+8} \right\} = 3\cos(2\sqrt{2}t) - \dfrac{12}{2\sqrt{2}} \sin(2\sqrt{2}t)$

$$= 3\cos(2\sqrt{2}t) - 3\sqrt{2}\sin(2\sqrt{2}t)$$

34. $\mathcal{L}^{-1}\left\{ \left(\dfrac{\sqrt{s}-1}{s}\right)^2 \right\} = \mathcal{L}^{-1}\left\{ \dfrac{s+1-2\sqrt{s}}{s^2} \right\}$

$$= \mathcal{L}^{-1}\left\{ \frac{1}{s} + \frac{1}{s^2} - \frac{2}{s^{3/2}} \right\}$$

$$= \frac{1}{\Gamma(1)} t^0 + \frac{1}{\Gamma(2)} t^1 - \frac{2}{\Gamma(3/2)} t^{1/2}$$

$$= 1 + t - \frac{2\sqrt{t}}{1/2\sqrt{\pi}} = 1 + t - 4\sqrt{\frac{t}{\pi}}$$

35. $\mathcal{L}^{-1}\left\{ \frac{1}{s} \sin\left(\frac{1}{s}\right) \right\}$

$$= \mathcal{L}^{-1}\left\{ \frac{1}{s}\left(\frac{1}{s} - \frac{(1/s)^3}{3!} + \frac{(1/s)^5}{5!} - \cdots \right)\right.$$

$$= \mathcal{L}^{-1}\left\{ \frac{1}{s^2} - \frac{(1/s)^4}{3!} + \frac{(1/s)^6}{5!} - \cdots \right)$$

$$= \frac{1}{1!}t^1 - \frac{1}{3!3!}t^3 + \frac{1}{5!5!}t^5 - \dots$$

$$= \sum_{n=1}^{\infty} \frac{t^{2n-1}}{((2n-1)!)^2}(-1)^{n+1}$$

36. $\mathcal{L}^{-1}\left\{\tan^{-1}\left(\frac{2}{s^2}\right)\right\} = \frac{-1}{t}\mathcal{L}^{-1}\left\{\frac{1}{1+\left(\frac{2}{s^2}\right)^2} \cdot \frac{(2)(-2)}{s^3}\right\}$

$$= \frac{-1}{t}\mathcal{L}^{-1}\left\{\frac{-4s}{(s^2)^2+2^2}\right\} = \frac{-1}{t}\mathcal{L}^{-1}\left\{\frac{-4s}{s^4+4+4s^2-4s^2}\right\}$$

$$= \frac{-1}{t}\mathcal{L}^{-1}\left\{\frac{-4s}{(s^2+2)^2-(2s)^2}\right\} = \frac{-1}{t}\mathcal{L}^{-1}\left\{\frac{-4s}{(s^2-2s+2)(s^2+2s+2)}\right\}$$

$$= \frac{-1}{t}\mathcal{L}^{-1}\left\{\frac{(s^2-2s+2)-(s^2+2s+2)}{(s^2-2s+2)(s^2+2s+2)}\right\}$$

$$= \frac{-1}{t}\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s+2} - \frac{1}{s^2-2s+2}\right\}$$

$$= \frac{-1}{t}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1} - \frac{1}{(s-1)^2+1}\right\}$$

$$= \frac{-1}{t}\left(e^{-t}\sin t - e^{t}\sin t\right)$$

$$= +\frac{2\sin t}{t}\sinh t$$

37. $\mathcal{L}^{-1}\left\{ \ln\left(1-\frac{a^2}{s^2}\right)\right\} = \mathcal{L}^{-1}\left\{ \ln(s^2-a^2) - 2\ln s\right\}$

$= \frac{-1}{t}\mathcal{L}^{-1}\left\{ \frac{2s}{s^2-a^2} - \frac{2}{s}\right\}$

$= \frac{-1}{t}\left(2\cosh at - 2\right)$

38. $\mathcal{L}^{-1}\left\{ \cot^{-1}\left(\frac{s+3}{2}\right)\right\}$

$= +\frac{1}{t}\mathcal{L}^{-1}\left\{ \frac{1}{1+\frac{(s+3)^2}{2^2}} \cdot \frac{1}{2}\right\} = \frac{1}{t}\mathcal{L}^{-1}\left\{\frac{2}{(s+3)^2+2^2}\right\}$

$= \frac{e^{-3t}}{t}\sin 2t$

39. $\mathcal{L}^{-1}\left\{ \frac{s}{(s^2+1)^2}\right\} = -\frac{1}{2}\mathcal{L}^{-1}\left\{ \frac{-2s}{(s^2+1)^2}\right\} = -\frac{1}{2}\mathcal{L}^{-1}\left\{ \frac{d}{ds}\left(\frac{1}{s^2+1}\right)\right\}$

$= +\frac{t}{2}\sin t$

40. $\mathcal{L}^{-1}\left\{ \frac{1}{s^2}\left[\frac{s+1}{s^2+1}\right]\right\} = \int_0^t\int_0^t \mathcal{L}^{-1}\left\{ \frac{s+1}{s^2+1}\right\} dt\, dt$

$= \int_0^t\int_0^t \cos t + \sin t \, dt\, dt = \int_0^t \left[\sin t - \cos t\right]_0^t dt$

$$= \int_0^t \sin t - \cos t + 1 \, dt = \left[-\cos t - \sin t + t \right]_0^t$$

$$= -\cos t + 1 - \sin t + t$$

41. $$\mathcal{L}^{-1}\left\{ \frac{1}{s(s^2+a^2)} \right\} = \int_0^t \mathcal{L}^{-1}\left\{ \frac{1}{s^2+a^2} \right\} dt$$

$$= \int_0^t \frac{1}{a} \sin at \, dt = \frac{-1}{a^2}\left[\cos at \right]_0^t = \frac{1}{a^2}\left(1 - \cos at \right)$$

42. $$\mathcal{L}^{-1}\left\{ \frac{e^{-4s} - e^{-7s}}{s^2} \right\} = \frac{u(t-4)(t-4)}{1} - \frac{u(t-7)(t-7)}{1}$$

$$= (t-4)\,u(t-4) - (t-7)\,u(t-7) \qquad -t+7 = x - t + 4$$

$$= \overset{f_1}{0} + (\overset{f_2 - f_1}{t-4-0})\,u(t-4) + (\overset{f_3}{3} - \overset{f_2}{(t-4)})\,u(t-7)$$

$$f(t) = \begin{cases} 0, & 0 < t < 4 \\ t-4, & 4 < t < 7 \\ 3, & t \geq 7 \end{cases}$$

43. $$\mathcal{L}^{-1}\left\{ \frac{3}{s} - \frac{4e^{-s}}{s^2} + \frac{4e^{-3s}}{s^2} \right\}$$

$$= 3 - 4(u-1)(t-1) + 4(u-3)(t-3)$$

$$= 3 + (u-1)((-4t+7)-3) + (u-3)($$

$$= \begin{cases} 3, & 0 < t < 1 \\ 7-4t, & 1 \leq t < 3 \\ -5, & t \geq 3 \end{cases}$$

44. $$\mathcal{L}^{-1}\left\{ \frac{s^2+9s-9}{s(s^2-9)} \right\} = \mathcal{L}^{-1}\left\{ \frac{s^2-9}{s(s^2-9)} + \frac{9s}{s(s^2-9)} \right\}$$

$$= 1 + \mathcal{L}^{-1}\left\{ \frac{9}{s^2-9} \right\} = 1 + 3\sinh 3t$$

45. $$\mathcal{L}^{-1}\left\{ \frac{s+1}{(s^2+1)(s^2+4)} \right\}$$

$$\frac{s+1}{(s^2+1)(s^2+4)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4}$$

$$s+1 = (As+B)(s^2+4) + (Cs+D)(s^2+1)$$

$s = 2i$,

$$2i+1 = (2Ci+D)(-3)$$
$$2i+1 = -6Ci - 3D$$

$$C = -1/3 \qquad D = -1/3$$

$s = i$

$$1+i = (Ai+B)(3)$$

$$A = 1/3 \qquad B = 1/3$$

$$= \mathcal{L}^{-1}\left\{ \frac{s+1}{3(s^2+1)} - \frac{(s+1)}{3(s^2+4)} \right\}$$

$$= \frac{1}{3}\left(\cos t + \sin t - \cos 2t + \frac{1}{2}\sin 2t \right)$$

Convolution Theorem

Definition of Convolution

The convolution of two functions f(t) and g(t), denoted by $f(t) * g(t)$ or $(f*g)t$ is defined as

$$f(t) * g(t) = \int_0^t f(u)g(t-u)\,du$$

Convolution Theorem

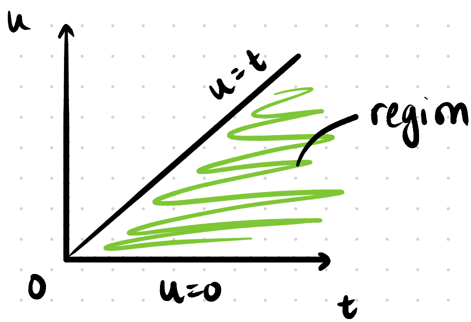
If $\mathcal{L}^{-1}\{F(s)\} = f(t)$ and $\mathcal{L}^{-1}\{G(s)\} = g(t)$, then

$$\mathcal{L}^{-1}\{F(s)G(s)\} = \int_0^t f(u)g(t-u)\,du = f(t) * g(t)$$

Proof

By the definition of LT,

$$\mathcal{L}\{f*g\} = \int_0^\infty e^{-st}\left(\int_0^t f(u)g(t-u)\,du\right)dt$$



Changing the order of integration

$$\mathcal{L}\{f*g\} = \int_{u=0}^{\infty} \int_{t=u}^{\infty} e^{-st} (f(u)\, g(t-u))\, dt\, du$$

let $v = t - u$ $\qquad t = u \,;\; v = 0$

$dv = dt$ $\qquad t = \infty \,;\; v = \infty$

$$\mathcal{L}\{f*g\} = \int_{u=0}^{\infty} \int_{v=0}^{\infty} e^{-s(v+u)} (f(u)\, g(v))\, dv\, du$$

$$= \int_0^{\infty} e^{-su} f(u)\, du \int_0^{\infty} e^{-sv} g(v)\, dv$$

$$= \mathcal{L}\{f(u)\}\, \mathcal{L}\{g(v)\}$$

$$\mathcal{L}\{f*g\} = F(s)\, G(s)$$

$$\Rightarrow \mathcal{L}^{-1}\{F(s)\, G(s)\} = f*g$$

Note: take $G(s)$ to be an easy function for integration

Use convolution theorem to find inverse Laplace of the following.

46. $\mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s+9)^2}\right\}$

Let $F(s) = \frac{1}{(s+9)^2}$ and $G(s) = \frac{1}{s+1}$

$f(t) = e^{-9t}t$ $\qquad g(t) = e^{-t}$

$$\mathcal{L}^{-1}\{F(s)G(s)\} = (f*g)t = \int_0^t f(u)g(t-u)\,du$$

$$= \int_0^t e^{-9u}u\, e^{-(t-u)}\,du$$

$$= \int_0^t u\, e^{-8u} e^{-t}\,du = e^{-t}\int_0^t u\, e^{-8u}\,du$$

$s = u$ $\qquad v = -\frac{1}{8}e^{-8u}$

$ds = du$ $\qquad dv = e^{-8u}\,du$

$$= \left[\frac{-ue^{-8u}}{8} + \frac{1}{8}\int_0^t e^{-8u}\,du\right]_0^t e^{-t}$$

$$= \left(\frac{-te^{-8t}}{8} - \frac{1}{64}e^{-8t} + \frac{1}{64}\right)e^{-t} = \frac{1}{64}\left(e^{-t} - e^{-9t} - 8te^{-9t}\right)$$

47. $\mathcal{L}^{-1}\left\{\frac{s^2}{(s^2+a^2)^2}\right\}$

Let $F(s) = \frac{s}{s^2+a^2}$ $\qquad$ $G(s) = \frac{s}{s^2+a^2}$

$f(t) = \cos at$ $\qquad$ $g(t) = \cos at$

$$\mathcal{L}^{-1}\{F(s)G(s)\} = \int_0^t \cos au \cos(at-au)\, du$$

$$= \frac{1}{2}\int_0^t \cos at + \cos(2au-at)\, du$$

$$= \frac{1}{2}\left[u\cos at + \frac{\sin(2au-at)}{2a} \right]_0^t$$

$$= \frac{1}{2}\left(t\cos at + \frac{\sin at}{2a} + \frac{\sin at}{2a} \right)$$

$$= \frac{1}{2}\left(t\cos at + \frac{\sin at}{a} \right)$$

48. $\mathcal{L}^{-1}\left\{\frac{1}{(s^2+4s+13)^2}\right\}$

$F(s) = \frac{1}{s^2+4s+13}$ $\qquad$ $G(s) = \frac{1}{s^2+4s+13}$

$$F(s) = \frac{1}{(s+2)^2+3^2} \qquad G(s) = \frac{1}{(s+2)^2+3^2}$$

$$f(t) = \frac{e^{-2t}}{3}\sin 3t \qquad g(t) = \frac{e^{-2t}}{3}\sin 3t$$

$$\mathcal{L}^{-1}\{F(s)G(s)\} = \int_0^t \left(\frac{e^{-2u}}{3}\sin 3u\right)\left(\frac{e^{-2(t-u)}}{3}\sin 3(t-u)\right) du$$

$$= \frac{e^{-2t}}{9}\int_0^t \sin 3u \sin(3t-3u)\, du$$

$$= \frac{e^{-2t}}{18}\int_0^t \cos(6u-3t) - \cos 3t \, du$$

$$= \frac{e^{-2t}}{18}\left[\frac{\sin(6u-3t)}{6} - u\cos 3t\right]_0^t$$

$$= \frac{e^{-2t}}{18}\left(\frac{\sin 3t}{3} - t\cos 3t\right)$$

49. Verify Convolution Theorem for the functions

$f(t) = t$ and $g(t) = \cos t$

$$\mathcal{L}\{(f*g)t\} = \mathcal{L}\{f(t)\}\,\mathcal{L}\{g(t)\}$$

$$(f*g)t = \mathcal{L}^{-1}\{F(s)\,G(s)\}$$

LHS:

$$(f*g)t = \int_0^t u\cos(u-t)\,du$$

$s = u$ $ds = du$ $v = \sin(u-t)$ $dv = \cos(u-t)\,du$

$$= \left[u\sin(u-t) - \int \sin(u-t)\,du\right]_0^t$$

$$= +\left[\cos(u-t)\right]_0^t = 1 - \cos t \longrightarrow (1)$$

RHS:

$$F(s) = \frac{1}{s^2} \qquad G(s) = \frac{s}{s^2+1}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\left(\frac{1}{s^2+1}\right)\right\} = \int_0^t \sin t\,dt$$

$$= \left[-\cos t\right]_0^t = 1 - \cos t \longrightarrow (2)$$

$(1) = (2) \Rightarrow$ LHS = RHS

50. $\mathcal{L}^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\}$

$$F(s) = \frac{s}{s^2+a^2} \qquad G(s) = \frac{s}{s^2+b^2}$$

$$f(t) = \cos at \qquad g(t) = \cos bt$$

$$\mathcal{L}^{-1}\{F(s)G(s)\} = \int_0^t \cos au \cos(bt-bu)\,du$$

$$= \frac{1}{2}\int_0^t \cos(bt+u(a-b)) + \cos((a+b)u-bt)\,du$$

$$= \frac{1}{2}\left[\frac{\sin(bt+u(a-b))}{a-b} + \frac{\sin((a+b)u-bt)}{a+b}\right]_0^t$$

$$= \frac{1}{2}\left(\frac{\sin at}{a-b} - \frac{\sin bt}{a-b} + \frac{\sin at}{a+b} + \frac{\sin bt}{a+b}\right)$$

$$= \frac{1}{2}\left(\sin at\left(\frac{1}{a-b}+\frac{1}{a+b}\right) + \sin bt\left(\frac{1}{a+b}-\frac{1}{a-b}\right)\right)$$

$$= \frac{1}{2}\left(\frac{2a\sin at}{a^2-b^2} + \frac{2b\sin bt}{a^2-b^2}\right)$$

$$= \frac{a\sin at + b\sin bt}{a^2-b^2}$$

51. $\mathcal{L}^{-1}\left\{ \frac{4s+5}{(s-1)^2(s+2)} \right\}$

$F(s) = \frac{1}{s+2}$ $\qquad G(s) = \frac{4s+5}{(s-1)^2} = \frac{4(s-1)+9}{(s-1)^2}$

$f(t) = e^{-2t}$

$g(t) = 4e^{t} + 9e^{t}t = e^{t}(4+9t)$

$$\mathcal{L}^{-1}\{F(s)G(s)\} = \int_0^t f(u)\,g(t-u)\,du$$

$$= \int_0^t e^{-2u}\, e^{t-u}(4+9t-9u)\,du$$

$$= e^{t}\int_0^t e^{-3u}\left((4+9t)-9u\right)du$$

$$= (4+9t)e^{t}\int_0^t e^{-3u}\,du - 9e^{t}\int_0^t u\,e^{-3u}\,du$$

$u = u$ $\quad v = \frac{-1}{3}e^{-3u}$

$du = du$ $\quad dv = e^{-3u}$

$$= \frac{(4+9t)e^{t}}{-3}\left[e^{-3u}\right]_0^t - 9e^{t}\left[-\frac{e^{-3u}u}{3} - \frac{1}{9}e^{-3u}\right]_0^t$$

$$= -\frac{(4+9t)e^{t}}{3}\left(e^{-3t}-1\right) - 9e^{t}\left(\frac{1}{9} - \frac{te^{-3t}}{3} - \frac{e^{-3t}}{9}\right)$$

$$= e^{t}\left(-\frac{4}{3}e^{-3t} - 3\cancel{te^{-3t}} + \frac{4}{3} + 3t - 1 + 3\cancel{te^{-3t}} + e^{-3t}\right)$$

$$= e^{t}\left(-\frac{1}{3}e^{-3t} + \frac{1}{3} + 3t\right)$$

52. $\mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s^2+1)}\right\}$

let $F = \frac{1}{s+1}$ $\quad G = \frac{1}{s^2+1}$

$f = e^{-t}$ $\quad g = \sin t$

OR

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$$

$$\mathcal{L}^{-1}\{F\cdot G\} = \int_0^t e^{-u}\sin(t-u)\,du$$

$$= \operatorname{Im}\left(\int_0^t e^{-u}e^{i(t-u)}\,du\right) = \operatorname{Im}\left(e^{it}\int_0^t e^{-u(1+i)}\,du\right)$$

$$= \operatorname{Im}\left(e^{it}\cdot\left[\frac{e^{-u(1+i)}}{-(1+i)}\right]_0^t\right)$$

$$= \operatorname{Im}\left(-e^{it}\frac{(e^{-t(1+i)}-1)(1-i)}{(1+i)(1-i)}\right)$$

$$= \operatorname{Im}\left(-\frac{1}{2}\left(e^{it}\left(e^{-t-it}-1-ie^{-t-it}+i\right)\right)\right)$$

$$= -\frac{1}{2}\operatorname{Im}\left(e^{-t}-e^{it}-ie^{-t}+ie^{it}\right)$$

$$= -\frac{1}{2}\operatorname{Im}\left(e^{-t}-\cos t - i\sin t - ie^{-t} + i\cos t - \sin t\right)$$

$$= -\frac{1}{2}\left(-\sin t - e^{-t} + \cos t\right)$$

53. $\mathcal{L}^{-1}\left\{\frac{1}{(s^2+4)(s+1)^2}\right\}$

$F(s) = \frac{1}{s^2+4}$ $\qquad$ $G(s) = \frac{1}{(s+1)^2}$

$f(t) = \frac{1}{2}\sin 2t$ $\qquad$ $g(t) = e^{-t}\, t$

$$\mathcal{L}^{-1}\{F(s)\,G(s)\} = \int_0^t \frac{\sin 2u}{2}\left(e^{-t+u}(t-u)\right)du$$

$$= \frac{1}{2}\int_0^t \sin 2u \; e^{-(t-u)}\,(t-u)\,du$$

$$= \frac{1}{2}\,\mathrm{Im}\left(\int_0^t \underbrace{e^{i2u-t+u}}_{dv}\,\underbrace{(t-u)}_{u}\,du\right) \qquad u = t-u \quad du = -du \qquad v = \frac{e^{u(1+2i)-t}}{1+2i} \quad dv = e^{u(1+2i)-t}$$

$$= \frac{1}{2}\,\mathrm{Im}\left(\left[\frac{(t-u)\,e^{u(1+2i)-t}}{1+2i}\right]_0^t + \int_0^t \frac{e^{u(1+2i)-t}}{1+2i}\,du\right)$$

$$= \frac{1}{2}\,\mathrm{Im}\left(\frac{-te^{-t}}{1+2i} + \left[\frac{e^{u(1+2i)-t}}{(1+2i)(1+2i)}\right]_0^t\right)$$

$$= \frac{1}{2}\,\mathrm{Im}\left(\frac{-t(1-2i)e^{-t}}{5} + \frac{(1-2i)^2}{25}\left(e^{2ti} - e^{-t}\right)\right)$$

$$= \frac{1}{2}\,\mathrm{Im}\left(\frac{-te^{-t}}{5} + \frac{i2te^{-t}}{5} + \left(\frac{(1-4)}{25} - \frac{4i}{25}\right)\left(\cos 2t + i\sin 2t - e^{-t}\right)\right)$$

$$= \frac{1}{2}\left(\frac{2te^{-t}}{5} + \frac{-3}{25}\sin 2t - \frac{4\cos 2t}{25} + \frac{4e^{-t}}{25} \right)$$

$$= \frac{1}{2}\left(\frac{10te^{-t} - 3\sin 2t - 4\cos 2t + 4e^{-t}}{25} \right)$$

$$= \frac{1}{50}\left(10te^{-t} - 3\sin 2t - 4\cos 2t + 4e^{-t} \right)$$

54. $\mathcal{L}^{-1}\left\{ \frac{1}{(s+2)^2(s-2)} \right\}$

$$F(s) = \frac{1}{(s+2)^2} \Rightarrow f(t) = e^{-2t}t \; ; \; G(s) = \frac{1}{s-2} \rightarrow g(t) = e^{2t}$$

$$\mathcal{L}^{-1}\{F(s)G(s)\} = \int_0^t e^{-2u}u \; e^{2t-2u} \, du$$

$$= e^{2t}\int_0^t e^{-4u} u \, du = e^{2t}\left[\frac{ue^{-4u}}{-4} - \int \frac{e^{-4u}}{-4} \, du \right]_0^t$$

$$= e^{2t}\left(\frac{te^{-4t}}{-4} - \frac{1}{16}e^{-4t} + \frac{1}{16} \right)$$

$$= \frac{1}{16}\left(-4e^{-2t} - e^{-2t} + e^{2t} \right)$$

Solution to Differential Equations by Laplace Transforms

Procedure

Step I: Take LT on both sides

Step II: Convert LT eq. to an algebraic eq. using LT of derivatives and boundary conditions

Step III: By grouping, find $y(t)$

Formulas

$$\mathcal{L}\{y'(t)\} = sF(s) - y(0)$$

$$\mathcal{L}\{y''(t)\} = s^2F(s) - sy(0) - y'(0)$$

$$\mathcal{L}\{y'''(t)\} = s^3F(s) - s^2y(0) - sy'(0) - y''(0)$$

$$\mathcal{L}\{y''''(t)\} = s^4F(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0)$$

where $F(s) = \mathcal{L}\{f(t)\}$

55. Solve the DE using LT

$$y'' - 3y' + 2y = 12e^{-2t}, \quad y(0) = 2, \quad y'(0) = 6$$

Taking LT on both sides,

$$\mathcal{L}\{y''(t)\} - 3\mathcal{L}\{y'(t)\} + 2\mathcal{L}\{y(t)\} = 12\mathcal{L}\{e^{-2t}\}$$

Let $F(s) = \mathcal{L}\{y(t)\}$

$$s^2F(s) - sy(0) - y'(0) - 3(sF(s) - y(0)) + 2F(s) = \frac{12}{s+2}$$

$$s^2F - 2s - 6 - 3(sF - 2) + 2F = \frac{12}{s+2}$$

$$F(s^2 - 3s + 2) - 2s - \cancel{6} + \cancel{6} = \frac{12}{s+2}$$

$$F(s^2 - 3s + 2) = \frac{12}{s+2} + 2s$$

$$F(s) = \frac{12}{(s-1)(s-2)(s+2)} + \frac{2s}{(s-1)(s-2)}$$

$$= \frac{12}{(s-1)(s^2-4)} + \frac{2s}{(s-1)(s-2)}$$

Taking inverse LT on both sides

$$y(t) = \mathcal{L}^{-1}\left\{\frac{12}{(s-1)(s^2-4)}\right\} + \mathcal{L}^{-1}\left\{\frac{2s}{(s-1)(s-2)}\right\}$$

(I) $\mathcal{L}^{-1}\left\{ \frac{12}{(s-1)(s-2)(s+2)} \right\}$

Using partial fractions

$$\frac{12}{(s-1)(s-2)(s+2)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s+2}$$

$$12 = A(s-2)(s+2) + B(s-1)(s+2) + C(s-2)(s-1)$$

$s=1$,

$12 = -3A \Rightarrow A = -4$

$s=2$,

$12 = 4B \Rightarrow B = 3$

$s=-2$,

$12 = 12C \Rightarrow C = 1$

$$= \frac{-4}{s-1} + \frac{3}{s-2} + \frac{1}{s+2}$$

$$\mathcal{L}^{-1}\left\{ \frac{-4}{s-1} + \frac{3}{s-2} + \frac{1}{s+2} \right\} = -4e^{t} + 3e^{2t} + e^{-2t}$$

(II) $\mathcal{L}^{-1}\left\{ \frac{2s}{(s-1)(s-2)} \right\}$

Partial fractions

$$\frac{2s}{(s-1)(s-2)} = \frac{A}{s-1} + \frac{B}{s-2}$$

$$2s = A(s-2) + B(s-1)$$

$s=2$,

$$4 = B \Rightarrow B = 4$$

$s=1$,

$$2 = -A \Rightarrow A = -2$$

$$= \mathcal{L}^{-1}\left\{ \frac{-2}{s-1} + \frac{4}{s-2} \right\}$$

$$= -2e^{t} + 4e^{2t}$$

$$\boxed{y(t) = -6e^{t} + 7e^{2t} + e^{-2t}}$$

56. $y'' + y = f(t)$; $y(0) = 1$, $y'(0) = 0$

$$f(t) = \begin{cases} 3 , & 0 \le t \le 4 \\ 2t-5 , & t > 4 \end{cases}$$

$$f(t) = 3 + u(t-4)((2t-5)-3)$$

Taking LT on both sides

$$s^2 \mathcal{L}\{y(t)\} - s y(0) - y'(0) + \mathcal{L}\{y(t)\} = \frac{3}{s} + \mathcal{L}\{u(t-4)(2t-8)\}$$

$$\mathcal{L}\{y(t)\}(s^2+1) - s = \frac{3}{s} + \frac{2e^{-4s}}{s^2}$$

$$\mathcal{L}\{y(t)\} = \frac{3}{s(s^2+1)} + \frac{2e^{-4s}}{s^2(s^2+1)} + \frac{s}{s^2+1}$$

Taking inverse LT

$$y(t) = 3\int_0^t \sin t \, dt + \mathcal{L}^{-1}\left\{\frac{2e^{-4s}((s^2+1)-s^2)}{s^2(s^2+1)}\right\} + \cos t$$

$$= 3(1-\cos t) + \cos t + \mathcal{L}^{-1}\left\{\frac{2e^{-4s}}{s^2} - \frac{2e^{-4s}}{s^2+1}\right\}$$

$$y(t) = 3 - 2\cos t + 2u(t-4)(t-4) - 2u(t-4)(\sin(t-4))$$

$$= (3 - 2\cos t) + u(t-4)(2t - 8 - 2\sin(t-4))$$

$$2t - 8 - 2\sin(t-4) = x - 3 + 2\cos t$$

$$x = 2t - 8 - 2\sin(t-4) + 3 - 2\cos t$$

$$= 2t - 5 - 2\cos t - 2\sin(t-4)$$

$$y(t) = \begin{cases} 3 - 2\cos t, & 0 \le t \le 4 \\ 2t - 5 - 2\cos t - 2\sin(t-4), & t > 4 \end{cases}$$

56. $\frac{dy}{dt} + 3y + 2\int_0^t y\,dt = t, \quad y(0) = 0$

$$\mathcal{L}\{y'(t)\} + 3\mathcal{L}\{y(t)\} + \frac{2}{s}\mathcal{L}\{y(t)\} = \frac{1}{s^2}$$

$$s\mathcal{L}\{y(t)\} - 0 + 3\mathcal{L}\{y(t)\} + \frac{2}{s}\mathcal{L}\{y(t)\} = \frac{1}{s^2}$$

$$\mathcal{L}\{y(t)\}\left(\frac{s^2 + 3s + 2}{\cancel{s}}\right) = \frac{1}{s^{\cancel{2}}}$$

$$\mathcal{L}\{y(t)\} = \frac{1}{s(s+2)(s+1)}$$

$$y(t) = \mathcal{L}^{-1}\left\{ \frac{1}{s(s+2)(s+1)} \right\}$$

$$\frac{1}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

$$1 = A(s+1)(s+2) + B(s)(s+2) + C(s)(s+1)$$

$s = 0$,

$$1 = 2A \Rightarrow A = 1/2$$

$s = -1$

$$1 = -B \Rightarrow B = -1$$

$s = -2$,

$$1 = 2C \Rightarrow C = 1/2$$

$$= \mathcal{L}^{-1}\left\{ \frac{1/2}{s} - \frac{1}{s+1} + \frac{1/2}{s+2} \right\}$$

$$y(t) = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}$$

(not sure)

A 51. $ty'' - (2+t)y' + 3y = t-1$, $y(0) = 0$

since $y'(0)$ not given, we assume $y'(0) = a$

Taking LT,

$$(-1)^1 \frac{d}{ds}\left(s^2 Y(s) - s y(0) - y'(0)\right) - 2\left(sY(s) - y(0)\right)$$

$$+ \frac{d}{ds}\left(sY(s) - y(0)\right) + 3Y(s) = \frac{1}{s^2} - \frac{1}{s}$$

$$-\left(2sY(s) + s^2 Y'(s)\right) - 2sY(s) + \left(Y(s) + sY'(s)\right) + 3Y(s) = \frac{1}{s^2} - \frac{1}{s}$$

$$Y'(s)(-s^2+s) + Y(s)(-2s - 2s + 1 + 3) = \frac{1}{s^2} - \frac{1}{s}$$

$$-Y'(s)(s)(s-1) + Y(s)(-4)(s-1) = \frac{1-s}{s^2}$$

$$sY'(s) + 4Y(s) = \frac{1}{s^2}$$

$$Y'(s) + \frac{4}{s}Y(s) = \frac{1}{s^3}$$

$$\frac{dy}{ds} + Py = Q$$

$$IF = e^{\int P ds} = e^{\int \frac{4}{s} ds} = e^{4 \ln s} = s^4$$

$$Y(s)\, s^4 = \int \frac{s^4}{s^3} ds = \frac{s^2}{2} + c$$

$$Y(s) = \frac{1}{2s^2} + \frac{c}{s^4}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{2s^2} + \frac{c}{s^4}\right\}$$

$$y(t) = \frac{t}{2} + \frac{ct^3}{6}$$

58. $\frac{d^2y}{dt^2} + 9y = \cos 2t \quad , y(0)=1, \; y(\frac{\pi}{2}) = -1$

Since $y'(0)$ is not given, we assume $y'(0)=a$

Taking LT on both sides,

$$\mathcal{L}\{y''(t)\} + 9\mathcal{L}\{y(t)\} = \frac{s}{s^2+4}$$

$$s^2\mathcal{L}\{y(t)\} - sy(0) - y'(0) + 9\mathcal{L}\{y(t)\} = \frac{s}{s^2+4}$$

$$\mathcal{L}\{y(t)\}(s^2+9) - s - a = \frac{s}{s^2+4}$$

$$\mathcal{L}\{y(t)\} = \frac{s}{(s^2+4)(s^2+9)} + \frac{s}{s^2+9} + \frac{a}{s^2+9}$$

Taking inverse,

$$y(t) = \mathcal{L}^{-1}\left\{\left(\frac{s}{s^2+4}\right)\left(\frac{1}{s^2+9}\right)\right\} + \cos 3t + \frac{a}{3}\sin 3t$$

$F(s)$ $G(s)$

$f(t) = \cos 2t$

$g(t) = \frac{1}{3}\sin 3t$

$$\mathcal{L}^{-1}\{F(s)G(s)\} = \int_0^t \frac{1}{3}\cos 2u \sin(3t-3u)\, du$$

$$= \frac{1}{6}\int_0^t \sin(3t-u) + \sin(3t-5u)\, du$$

$$= \frac{1}{6}\left[\cos(3t-u) + \frac{1}{5}\cos(3t-5u)\right]_0^t$$

$$= \frac{1}{6}\left(\cos 2t + \frac{1}{5}\cos 2t - \cos 3t - \frac{1}{5}\cos 3t\right)$$

$$= \frac{1}{6}\left(\frac{6}{5}\cos 2t - \frac{6}{5}\cos 3t\right)$$

$$= \frac{1}{5}(\cos 2t - \cos 3t)$$

$$y(t) = \frac{4}{5}\cos 3t + \frac{1}{5}\cos 2t + \frac{a}{3}\sin 3t$$

$y(\frac{\pi}{2}) = -1$

$$-1 = \frac{4}{5}\cos\frac{3\pi}{2} + \frac{1}{5}\cos\pi + \frac{a}{3}\sin\frac{3\pi}{2}$$

0

$$-1 = -\frac{1}{5} - \frac{a}{3} \Rightarrow 1 = \frac{3+5a}{15} \Rightarrow \frac{12}{5} = a$$

$$y(t) = \frac{4}{5}\cos 3t + \frac{1}{5}\cos 2t + \frac{4}{5}\sin 3t$$

59. $y' - y = e^{3t}$, $y(0) = 2$

$$\mathcal{L}\{y'\} - \mathcal{L}\{y\} = \frac{1}{s-3}$$

$$s\mathcal{L}\{y\} - y(0) - \mathcal{L}\{y\} = \frac{1}{s-3}$$

$$\mathcal{L}\{y\}(s-1) = \frac{1}{s-3} + 2$$

$$\mathcal{L}\{y\} = \frac{1}{(s-3)(s-1)} + \frac{2}{s-1}$$

$$= \frac{1}{s^2-4s+3} + \frac{2}{s-1}$$

$$= \frac{1}{(s-2)^2-1} + \frac{2}{s-1}$$

Taking inverse,

$$y(t) = e^{2t}\sinh t + 2e^{t}$$

$$= e^{2t}\left(\frac{e^{t}-e^{-t}}{2}\right) + 2e^{t}$$

$$= \frac{e^{3t}}{2} - \frac{e^{t}}{2} + 2e^{t}$$

$$= \frac{e^{3t}}{2} + \frac{3}{2}e^{t}$$

60. $y'' - 6y' + 9y = 0$, $y(0) = 2$, $y'(0) = 9$

Taking LT

$$s^2\mathcal{L}\{y\} - sy(0) - y'(0) - 6s\mathcal{L}\{y\} + 6y(0) + 9\mathcal{L}\{y\} = \mathcal{L}\{0\}$$

$$\mathcal{L}\{y\}(s^2-6s+9) - 2s - 9 + 12 = 0$$

$$\mathcal{L}\{y\}(s^2-6s+9) = -3+2s = 2(s-3)+3$$

$$\mathcal{L}\{y\} = \frac{2(s-3)}{(s-3)^2} + \frac{3}{(s-3)^2}$$

Taking inverse

$$y(t) = 2e^{3t} + 3e^{3t}\,t$$
$$= e^{3t}(2+3t)$$

61. $y'' + y = e^{-2t}\sin t \quad , \; y(0)=0, \quad y'(0)=0$

Taking LT

$$s^2\mathcal{L}\{y\} - 0 - 0 + \mathcal{L}\{y\} = \frac{1}{(s+2)^2+1}$$

$$\mathcal{L}\{y\}\,(s^2+1) = \frac{1}{(s+2)^2+1}$$

$$\mathcal{L}\{y\} = \frac{1}{\underbrace{(s^2+1)}_{G(s)}\underbrace{((s+2)^2+1)}_{F(s)}}$$

$f(t) = e^{-2t}\sin t$
$g(t) = \sin t$

taking inverse,

$$y(t) = \int_0^t e^{-2u}\sin u\,\sin(t-u)\,du$$

$$= \frac{1}{2}\int_0^t e^{-2u}\left(\cos(2u-t) - \cos t\right)du$$

$$= \frac{1}{2}\int_0^t e^{-2u}\cos(2u-t) - \frac{1}{2}\cos t \int_0^t e^{-2u}\,du$$

$\int e^{ax}\cos bx$

$$= \left[\frac{1}{2}\left(\frac{e^{-2u}}{8}\right)\left(-2\cos(2u-t) + 2\sin(2u-t)\right)\right]_0^t$$

$$- \frac{1}{2}\cos t\left[\frac{e^{-2u}}{-2}\right]_0^t$$

$$= \frac{1}{2}\left(\frac{e^{-2t}}{8}(-2\cos t + 2\sin t) + (2\cos t + 2\sin t)\frac{1}{8}\right)$$

$$+ \frac{1}{4}\cos t\left(e^{-2t} - 1\right)$$

$$= \frac{e^{-2t}}{8}(\sin t + \cos t) + \frac{1}{8}(-\cos t + \sin t)$$

62. $y'''' - 16y = 30\sin t$

$y''(0) = 0, \quad y''(\pi) = 0$

$y'''(0) = -18, \quad y'''(\pi) = -18$

Taking LT,

a, b

$$s^4\mathcal{L}\{y\} - s^3 y(0) - s^2 y'(0) + 18 - 16\mathcal{L}\{y\} = 0$$

$$\mathcal{L}\{y\}(s^4-16) = s^3a + s^2b - 18$$

$$\mathcal{L}\{y\} = \frac{s^3a}{(s^2+4)(s^2-4)} + \frac{s^2b}{(s^2-4)(s^2+4)} - \frac{18}{(s^2-4)(s^2+4)}$$

$$\mathcal{L}\{y\} = \underbrace{\frac{s^3a}{(s^2+4)(s-2)(s+2)}}_{\text{I}} + \underbrace{\frac{s^2b}{(s^2+4)(s-2)(s+2)}}_{\text{II}} - \underbrace{\frac{18}{(s^2+4)(s-2)(s+2)}}_{\text{III}}$$

(I)

$$s^3a = (As+B)(s-2)(s+2) + C(s+2)(s^2+4) + D(s^2+4)(s-2)$$

$s=2$

$$\cancel{8}a = C(4)(\cancel{8}) \Rightarrow C = a/4$$

$s=-2$

$$+\cancel{8}a = D(\cancel{8})(+4) \Rightarrow D = a/4$$

$s=2i$

$$-\cancel{8}ia = (2Ai+B)(-\cancel{4}/4)$$

$$ia = 2Ai + B \Rightarrow A = a/2, \quad B = 0$$

$$= \frac{(a/2)s}{s^2+4} + \frac{a/4}{s-2} + \frac{a/4}{s+2}$$

(I)

$$s^2 b = (As+B)(s-2)(s+2) + C(s+2)(s^2+4) + D(s^2+4)(s-2)$$

$s = 2$

$$\cancel{4}b = C(\cancel{4})(8) \Rightarrow C = b/8$$

$s = -2$

$$\cancel{4}b = D(8)(-\cancel{4}) \Rightarrow D = -b/8$$

$s = 2i$

$$\cancel{-4}b = (\cancel{-8}^{\,2})(2Ai + B) \Rightarrow B = b/2, \ A = 0$$

$$= \frac{b/2}{s^2+4} + \frac{b/8}{s-2} + \frac{-b/8}{s+2}$$

(II)

$$-18 = (As+B)(s-2)(s+2) + C(s+2)(s^2+4) + D(s^2+4)(s-2)$$

$s = 2$

$$-\cancel{18}^{\,9} = C(\cancel{4}^{\,2})(8) \Rightarrow C = -9/16$$

$s = -2$

$$+18 = D(8)(+4) \Rightarrow D = 9/16$$

$s = 2i$,

$$-18 = (-8)(2Ai + B) \rightarrow A = 0,\ B = 9/4$$

$$= \frac{9/4}{s^2+4} + \frac{-9/16}{s-2} + \frac{9/16}{s+2}$$

Taking inverse,

$$y(t) = \frac{a}{2}\cos 2t + \frac{a}{4}e^{2t} + \frac{a}{4}e^{-2t} + \frac{b}{2}\cdot\frac{1}{2}\sin 2t$$

$$+ \frac{b}{8}e^{2t} - \frac{b}{8}e^{-2t}$$

$$+ \frac{9}{4}\cdot\frac{1}{2}\sin 2t - \frac{9}{16}e^{2t} + \frac{9}{16}e^{-2t}$$

$$= e^{2t}\left(\frac{b}{8} + \frac{a}{4} - \frac{9}{16}\right)$$